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The stability of plane Poiseuille flow between flexible walls

By C. H. GREEN† AND C. H. ELLEN
Imperial College, London

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This paper examines the linear stability of antisymmetric disturbances in incompressible plane Poiseuille flow between identical flexible walls which undergo transverse displacements. Using a variational approach, an approximate solution of the problem is formulated in a form suitable for computational evaluation of the (complex) wave speeds of the system. A feature of this formulation is that the varying boundary conditions (and the Orr–Sommerfeld equation) are satisfied only in the mean; this reduces the labour involved in determining the approximate solution for a variety of wall conditions without increasing the difficulty of obtaining solutions to a given accuracy. In this paper the symmetric stream function distribution across the channel is represented by a series of cosines whose coefficients are determined by the variational solution. Comparisons with previous work, both for the flexible-wall and rigid-wall problems, show that the method gives results as accurate as those obtained previously by other methods while new results, for flexible walls, indicate the presence of a higher wave-number stability boundary which joins the distorted Tollmien–Schlichting stability boundary at lower wave-numbers. In some cases this upper unstable region, which is characterized by large amplification rates, may determine the critical Reynolds number of the system.

1. Introduction

This paper represents a continuation of work by several authors on the effects of wall flexibility on the linear stability of parallel flows. Benjamin (1960) and Landahl (1962) have examined the question in some detail. In particular, Benjamin has presented an analysis which gives considerable insight into the problem and gives predictions for the beneficial design of flexible walls, although it should be kept in mind that the results may overlook certain unstable regions whose wave speeds are different from the Tollmien–Schlichting mode. One object of any further work in this field would be to examine the relevance of these concepts in situations where the limits on fluid–solid coupling are unrestricted. Further motivation, as for the rigid-wall case, is supplied by examining the question of the accuracy of the classical asymptotic theory on which the aforementioned analyses are based. Approximate methods, such as the step-by-step integration through the boundary layer performed by Hains & Price (1962) and

† Present address: The Nuclear Power Group, Radbroke Hall, Knutsford, Cheshire.

Landahl & Kaplan (1965), or 'direct methods' involving 'modal' series used for the rigid-wall problem by Lee & Reynolds (1967) and Grosch & Salwen (1968) give approximate, but highly accurate, results. One purpose of the present paper is to demonstrate the use of the modal method when the modes do not satisfy the boundary conditions of the problem and the errors in both the boundary conditions and the differential equation must be taken into account. There are a number of previous cases where the errors in the differential equation and boundary conditions, due to the substitution of an approximating series, have been combined (see, for example, the review article by Finlayson & Scriven 1966). Perhaps the most common case occurs in structural mechanics where a virtual work equation includes the effect of boundary loads but the virtual displacements are still required to satisfy the constraints or kinematic boundary conditions. If the problem has only 'natural' boundary conditions, or is self-adjoint, a way of combining the errors in the differential equation and boundary conditions can be chosen. The present problem does not fit into the above categories and it has been found useful to set up the problem in a vector space and use the adjoint system to obtain the approximate solution from a variational formulation, thus ensuring that the eigenvalue will be stationary with respect to variations in this approximate solution. In the present problem the boundary conditions are in general functions of the phase speed, which is employed as the eigenvalue. Hence the modes will not satisfy the boundary conditions unless chosen in a special and complicated way; instead, we chose modes which bear no relationship to the boundary conditions.

Previously published results have tended to concentrate, in the main, on the modification of the Tollmien–Schlichting stability curves by flexible walls, but it seems worthwhile to examine the stability of walls which are susceptible to other modes of instability and for which the stability curves may be composite. Wall flexibility, apart from changing the shape of the unstable region and perhaps introducing new unstable regions in the R, k plane, will also affect the wave speeds and amplification rates of the disturbances (see Benjamin 1964; Landahl & Kaplan 1965). These effects need to be considered in detail.

In this paper we limit ourselves to the case of plane Poiseuille flow although the method is equally applicable to the boundary-layer situation. This enables the non-dimensionalization to be based on constant quantities so that the wall equation will not be required to vary in the stream direction, a factor which may be important at low Reynolds numbers. Also the effect of three-dimensional disturbances may be gauged directly. Further, the basic flow profile is available in a simple analytic form and no ambiguity arises over its approximation. For a boundary-layer profile a high-order analytic approximation would be needed to give the integrals which arise in this method the same accuracy as that associated with the step length used in the numerical integration.

2. The stability problem

Linear perturbations of a two-dimensional incompressible parallel flow may be represented by a stream function $\phi(y)e^{ik(x-ct)}$ satisfying the Orr–Sommerfeld equation (see, for example, Lin 1955)

$$\phi^{iv} - 2k^2\phi'' + k^4\phi - ikR\{(U-c)(\phi''-k^2\phi) - U''\phi\} = 0, \quad (1)$$

where the variables are non-dimensionalized with respect to a reference length and flow velocity, and primes denote differentiation with respect to y, the co-ordinate normal to the basic flow $\mathrm{U}(y)$, which is in the x direction. For plane Poiseuille flow the reference length and velocity are conventionally the half-width of the channel and the maximum flow velocity, whence, for the x axis aligned with the centre-line of the channel, $U(y) = 1-y^2$. R is the Reynolds number and k, the streamwise wave-number, is associated with a disturbance whose phase speed is c (non-dimensionalized with the maximum flow velocity).

The boundary conditions for the problem are taken to be (see, for example, Hains & Price 1962)

$$\left.\begin{aligned} c\phi'(1) + U'(1)\phi(1) &= 0, \\ (E/R)\phi'''(1) + \phi(1) &= 0, \end{aligned}\right\} \quad (2)$$

and

$$\left.\begin{aligned} c\phi'(-1) + U'(-1)\phi(-1) &= 0, \\ (E/R)\phi'''(-1) - \phi(-1) &= 0, \end{aligned}\right\} \quad (3)$$

where the (assumed identical) wall properties are related by

$$\frac{R}{E} = -\left(mD + \frac{6}{c}\right)k^2 - \frac{imRk^3}{c}\left(\frac{c_W^2}{R^2} - c^2\right). \quad (4)$$

Here we confine our attention to walls which do not move parallel to themselves and which may be represented conveniently by parameters which have been non-dimensionalized with respect to the channel half-width, fluid density and kinematic viscosity (so that the Reynolds number may be changed through the flow velocity without affecting the wall properties). m is the ratio of the surface mass of the wall to the mass of fluid in a half-width channel; c_W, which may be a function of k, is the free-wave speed in an undamped wall and D is a wall viscous damping parameter, c_W and D non-dimensionalized with the fluid kinematic viscosity and the half-width of the channel. With this non-dimensionalization in the three-dimensional problem, oblique wave effects may be interpreted through changes in k alone with no change in wall properties being necessary. The expression for R/E takes into account the viscous component of the normal stress; if the viscous effect is ignored the second term in the right-hand side of (4) has a numerical constant 2 instead of 6 associated with it. In addition, since c and c_W have been non-dimensionalized in different ways, it will be convenient to introduce a further parameter $c_p = cR$ for comparison of the system wave speed with the wall free-wave speed.

We now propose to solve the boundary-value problem specified by (1), (2) and (3) approximately, with c as the eigenvalue and k as a real parameter. To do this a variational formulation of the problem is discussed first in general terms.

3. The general variational formulation

We consider the boundary-value problem in the symbolic form

$$(L_1 - cL_2)\phi = 0,$$

where L_1, L_2 are linear differential operators of order l, with boundary conditions

$$(M_s - cN_s)\phi_s = 0 \quad (s = 1, \dots, l),$$

where M_s, N_s are linear differential operators of order less than l.

Introducing a Hilbert space of complex vector functions quadratically integrable over the domain of the operators whose components are ϕ and the values that ϕ and its derivatives take on the boundaries, we rewrite the boundary-value problem as

$$\mathscr{L}\boldsymbol{\phi} = (\mathscr{L}_1\boldsymbol{\phi} - c\mathscr{L}_2\boldsymbol{\phi}) = \left\{ \begin{pmatrix} L_1\phi \\ M_1\phi_1 \\ \vdots \\ M_l\phi_l \end{pmatrix} - c \begin{pmatrix} L_2\phi \\ N_1\phi_1 \\ \vdots \\ N_l\phi_l \end{pmatrix} \right\} = \mathbf{0}, \tag{5}$$

where the first element is a function and the remaining elements are complex numbers.

For this space we define an inner product of two functions $\mathbf{f} = (f_1, \dots, f_n)$ and $\mathbf{g} = (g_1, \dots, g_n)$ by

$$\langle \mathbf{f}, \mathbf{g} \rangle = \frac{1}{V} \sum_{j=1}^{n} \int_V \bar{f}_j g_j \, dV, \tag{6}$$

where $\bar{\mathbf{f}}$ denotes the complex conjugate of $\mathbf{f}$ and V is the domain of definition of $\mathbf{f}$ and $\mathbf{g}$. For an operator $\mathscr{L}$ the adjoint operator $\mathscr{L}^*$, if it exists, may be defined by the relationship

$$\langle \boldsymbol{\phi}^*, \mathscr{L}\boldsymbol{\phi} \rangle - \langle \mathscr{L}^*\boldsymbol{\phi}^*, \boldsymbol{\phi} \rangle = 0, \tag{7}$$

where $\boldsymbol{\phi}$ and $\boldsymbol{\phi}^*$ are elements of the Hilbert space. As for scalar functions, with the stationary value of an integral generating the Euler equations of the problem, the stationary value of the functional $\langle \boldsymbol{\phi}^*, \mathscr{L}\boldsymbol{\phi} \rangle$ can be quite easily shown to generate the systems $\mathscr{L}\boldsymbol{\phi} = \mathbf{0}$ and $\mathscr{L}^*\boldsymbol{\phi}^* = \mathbf{0}$. Further, in the eigenvalue problem the eigenvalue is also stationary when the function is stationary and a minimum when $\mathscr{L} \equiv \mathscr{L}^*$, provided $\langle \boldsymbol{\phi}^*, \mathscr{L}_2\boldsymbol{\phi} \rangle \neq 0$.

Having presented this simple reformulation of the boundary-value problem we now follow the usual procedure to obtain a form for the approximate solution. Introducing sets of vector 'test functions' (or 'modes') $\boldsymbol{\Phi}_n$ and $\boldsymbol{\Phi}_n^*$, an approximation to $\boldsymbol{\phi}$ and $\boldsymbol{\phi}^*$ is made by substituting the finite series

$$\boldsymbol{\phi} = \sum_{n=1}^{N} a_n \boldsymbol{\Phi}_n, \quad \boldsymbol{\phi}^* = \sum_{n=1}^{N} \bar{a}_n^* \boldsymbol{\Phi}_n^* \tag{8}$$

(the complex conjugate of a_n^* being merely inserted for convenience) into either of the functionals $\langle \boldsymbol{\phi}^*, \mathscr{L}\boldsymbol{\phi} \rangle$ or $\langle \mathscr{L}^*\boldsymbol{\phi}^*, \boldsymbol{\phi} \rangle$, which is made stationary with respect to variations in each a_n, a_n^* in turn ($\partial/\partial a_n$, $\partial/\partial a_n^* = 0$, for $n = 1, \dots, N$). Thus we generate the following sets of linear algebraic equations in the a_n and a_n^*:

$$\left. \begin{aligned} \sum_{n=1}^{N} a_n \langle \boldsymbol{\Phi}_m^*, \mathscr{L}\boldsymbol{\Phi}_n \rangle = 0, \quad m = 1, \dots, N, \\ \sum_{n=1}^{N} a_n \langle \mathscr{L}^*\boldsymbol{\Phi}_m^*, \boldsymbol{\Phi}_n \rangle = 0, \quad m = 1, \dots, N, \end{aligned} \right\} \tag{9}$$

and

$$\left.\begin{aligned}\sum_{n=1}^{N} a_n^* \langle \mathbf{\Phi}_n^*, \mathscr{L}\mathbf{\Phi}_m\rangle &= 0, \quad m = 1, \ldots, N,\\ \sum_{n=1}^{N} a_n^* \langle \mathscr{L}^*\mathbf{\Phi}_n^*, \mathbf{\Phi}_m\rangle &= 0, \quad m = 1, \ldots, N.\end{aligned}\right\} \tag{10}$$

Regarding (9) and (10) as eigenvalue problems with eigenvectors whose components are the a_n, a_n^*, we may note that the matrices formed by the elements $\langle \mathbf{\Phi}_m^*, \mathscr{L}\mathbf{\Phi}_n\rangle$ and $\langle \mathscr{L}^*\mathbf{\Phi}_m^*, \mathbf{\Phi}_n\rangle$ are adjoint matrices and therefore, theoretically, have identical eigenvalues and eigenvectors. At the computational level it may be found that the matrices give different results owing to one being ill-conditioned (see Lee & Reynolds 1967). We also note that since the matrix elements in (10) are the transpose of the corresponding elements in (9) the eigenvalues will be the same for all equations; the eigenvectors, of course, will not. When $\mathbf{\Phi}_n^* \equiv \mathbf{\Phi}_n$, the system becomes a generalization of the equivalent Galerkin form for scalar equations. However, unlike that case, where the adjoint problem needed the same boundary conditions as its original problem (or the system was self-adjoint) before the Galerkin method was equivalent to a variational problem, here no such restriction exists. We may observe that if elements of the test function satisfy a particular boundary condition that boundary condition will not contribute to the inner product in (9) and (10); if all the boundary conditions are satisfied the problem reduces to the conventional scalar form.

The main point of this formulation is that we have now specified the way in which errors in the boundary conditions should be added into the error from the differential equation in order to make the eigenvalue stationary with respect to changes in the approximating series. The application to the plane Poiseuille problem is a demonstration of the use and accuracy of the method.

4. Approximate solution of the stability problem

We now write (1), (2) and (3) in the form specified by (5). However, for reasons associated with the algebraic complexity of the problem it has been found more satisfactory to increase the dimensions of the Hilbert space. Therefore we write

$$\mathscr{L}\boldsymbol{\phi} = \begin{bmatrix} \phi^{\text{iv}} - 2k^2\phi'' + k^4\phi - ikR\{U(\phi''-k^2\phi) - U''\phi\} \\ U'(1)\phi(1) \\ U'(-1)\phi(-1) \\ (E/R)\phi'''(1) + \phi(1) \\ (E/R)\phi'''(-1) - \phi(-1) \\ 0 \\ 0 \end{bmatrix} - c \begin{bmatrix} -ikR(\phi''-k^2\phi) \\ -\phi'(1) \\ -\phi'(-1) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}.$$

$\boldsymbol{\phi}$, $\boldsymbol{\phi}^*$ and $\mathscr{L}^*\boldsymbol{\phi}^*$ are now determined such that (7) is satisfied. Although the first components of $\boldsymbol{\phi}$, $\boldsymbol{\phi}^*$ and $\mathscr{L}^*\boldsymbol{\phi}^*$ must be 2ϕ, $2\phi^*$ and

$$\phi^{*\text{iv}} - 2k^2\phi^{*''} + k^4\phi^* + ikR\{(U\phi^*)'' - k^2U\phi^* - U''\phi^*\} + \bar{c}ikR(\phi^{*''} - k^2\phi^*),$$

respectively, because of (6) and (7), there is some flexibility in the choice of the other components. For further computation, we choose the adjoint system as

$$\mathscr{L}^*\boldsymbol{\phi}^* = \begin{bmatrix} \phi^{*\mathrm{iv}} - 2k^2\phi^{*''} + k^4\phi^* + ikR\{(U\phi^*)'' - k^2U\phi^* - U''\phi^*\} \\ \phi^{*'}(1) \\ \phi^{*'}(-1) \\ U'(1)\phi^{*''}(1) - 2U'(1)k^2\phi^*(1) \\ U'(-1)\phi^{*''}(-1) - 2U'(-1)k^2\phi^*(-1) \\ \phi^{*'}(1) \\ \phi^{*'}(-1) \end{bmatrix} - \bar{c}\begin{bmatrix} -ikR(\phi^{*''} - k^2\phi^*) \\ \phi^{*'}(1) \\ \phi^{*'}(-1) \\ -\phi^{*'''}(1) - (R/E)\phi^*(1) \\ -\phi^{*'''}(-1) + (R/E)\phi^*(-1) \\ 0 \\ 0 \end{bmatrix} = \mathbf{0},$$

and

$$\boldsymbol{\phi} = \begin{bmatrix} 2\phi \\ ikR\phi(1) \\ -ikR\phi(-1) \\ \phi'(1)/U'(1) \\ \phi'(-1)/U'(-1) \\ -\phi''(1) + (2k^2 - ikR)\phi(1) \\ \phi''(-1) - (2k^2 - ikR)\phi(-1) \end{bmatrix}, \quad \boldsymbol{\phi}^* = \begin{bmatrix} 2\phi^* \\ \dfrac{\phi^{*'''}(1)}{U'(1)} - \left(\dfrac{1}{U'(1)}\dfrac{R}{E} - ikR\right)\phi^*(1) \\ \dfrac{\phi^{*'''}(-1)}{U'(-1)} - \left(\dfrac{1}{U'(-1)}\dfrac{R}{E} - ikR\right)\phi^*(-1) \\ -\dfrac{R}{E}\phi^*(1) \\ \dfrac{R}{E}\phi^*(-1) \\ 0 \\ 0 \end{bmatrix}.$$

(For more details of this and other formulations of the problem, together with sample results comparing convergence, the reader is referred to Green (1970).) In effect, the adjoint differential equation together with its boundary conditions (bilinear concomitant set to zero) has been used to establish the structure of $\boldsymbol{\phi}$ and $\boldsymbol{\phi}^*$.

For an antisymmetric disturbance (symmetric stream function) the test functions can be defined by introducing cosine terms of the form $\cos(\frac{1}{2}n\pi y)$ for ϕ and ϕ^*. Therefore, $\mathbf{\Phi}_n$, $\mathbf{\Phi}_n^*$ are vectors formed by inserting $\cos(\frac{1}{2}n\pi y)$ into $\boldsymbol{\phi}$ and $\boldsymbol{\phi}^*$. These are substituted into (8) to generate (9) and (10). Thus we have, in matrix form,

$$(\mathbf{A} - c\mathbf{B})\,\mathbf{a} = \mathbf{0}, \tag{11}$$

$$(\mathbf{A}^T - c\mathbf{B}^T)\,\mathbf{a}^* = \mathbf{0}, \tag{12}$$

where

$$\mathbf{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix}, \quad \mathbf{a}^* = \begin{pmatrix} a_1^* \\ \vdots \\ a_N^* \end{pmatrix},$$

A and **B** are $N \times N$ matrices whose elements are defined by (9) and (10), and the superscript T denotes a transposed matrix. By performing the integrations involved in the inner products we may obtain expressions for general matrix elements of **A** and **B**. These reduce to simple algebraic forms (see Green 1970) which may be easily evaluated on a computer.

The computational procedure adopted was the following: we specify a given wall condition by specifying the parameters in (4) and a point in the R, k plane; an estimate of c is made so that we may evaluate E; now having fixed all the parameters in the matrices we evaluate the eigenvalue spectrum c. One of the eigenvalues in this spectrum should correspond to the initial guess; in general no eigenvalue from the spectrum will be sufficiently close to the initial guess and it is necessary to iterate to the correct result. Employing Kaplan's (1964) iterative scheme, it was found necessary to iterate only once or twice to obtain c correct to 4 decimal places, provided the initial estimate of c was reasonably good.

5. Results

For a given set of parameters, the spectra of c for the problem have been obtained from equation (11) using the adjoint operator, by first inverting **B** and using Share program SDA 3441 to obtain the eigenvalues of $\mathbf{AB}^{-1}$. The computations were performed in FORTRAN on the CDC 6600 of the University of London.

Comprehensive tests for a rigid wall have established the accuracy of the method (see Green 1970). For an examination of a flexible wall, the mass parameter m was fixed at a representative value of unity, while the damping parameter D was set at a value of 1000, to observe the effect of changing c_W (or tension, for a membrane). A value of $c_W = 10^5$ was found to simulate a rigid wall, with the stability curves insensitive to further increase in this parameter. Stability results were found for $c_W = 3000$, 5000, 10^4 and 10^5, which, for the value of D selected, imply the waves in the uncoupled wall are underdamped for the wave-number range of interest. The stability curves are shown in figure 1, where it can be seen that for the values $c_W = 3000$ and 5000 the neutral stability curves have two (joined) parts exhibiting different natures. These two parts will be referred to as the lower part and the upper part (from their position in the k versus R diagram). The neutral stability curve for $c_W = 10^4$ exhibited a similar characteristic, although its upper part was at a Reynolds number of roughly 15 000. Curves for larger values of c_W close to the rigid-wall curve (as typified by the curve at $c_W = 10^4$) have lower parts whose behaviour is predicted by Benjamin (1960) in his description of the class A mode. However, as c_W decreases further from its rigid-wall value, the damping starts to have a greater influence and the lower parts of the neutral stability curves become more unstable (in the sense of critical Reynolds number). The upper parts of the neutral stability curves also become steadily more unstable as the flexibility of the walls is increased (i.e. c_W decreases). A marked difference between the lower parts and the upper parts of the neutral stability curves is the different rate of amplification experienced when crossing the stability boundaries of these two parts. This property is illustrated by the dashed curve in figure 1, where a constant amplification curve calculated for $c_W = 5000$ is shown, representing a constant rate of amplification of $\mathrm{Im}\,(c_p) = 2{\cdot}5$. At $k = 0{\cdot}925$ on the lower part of the curve, it took an increase in R of roughly 400 to achieve this amplification rate, while at all points on the upper part of the curve the amplification rate was so rapid as to make distinction from the neutral curve impossible on this scale of the figure. In general, amplification rates increased by a factor of between 10 and 1000 for a given small increase

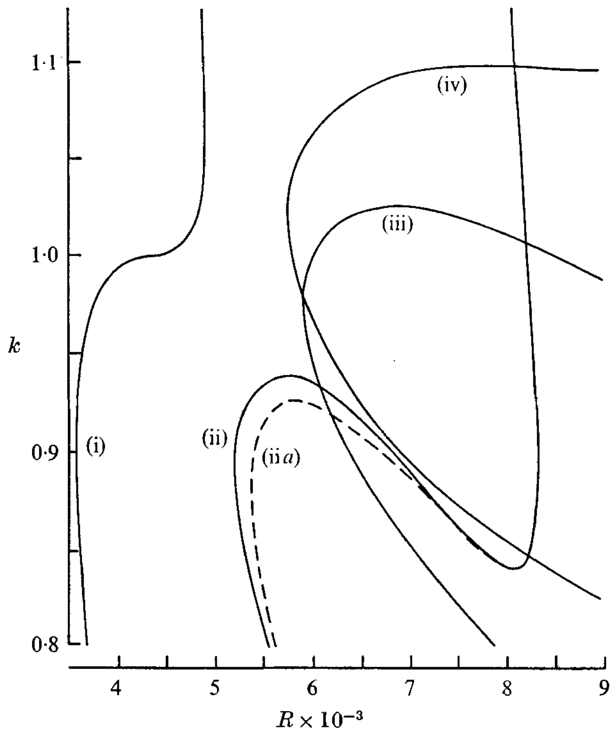

FIGURE 1. Neutral stability curves for damped membrane walls. Wave-number k versus Reynolds number R, for $m = 1$, $D = 1000$, $N = 30$.

	c_W	R_c	k_c
(i)	3000	3600	0·89
(ii)	5000	5270	0·89
(ii$a$)	5000: constant amplification curve, Im $(c_p) = 2{\cdot}5$.		
(iii)	10000	5940	0·97
(iv)	100000	5790	1·02 (an effectively rigid wall)

in R when moving from the lower to the upper parts of the stability curves. This difference between the magnitudes of the amplification rates experienced when crossing the two parts of the neutral stability curves is typified by a rapid change of slope in curves of $\mathrm{Im}\,(c_p)$ versus R (for fixed k) in the neighbourhood of $\mathrm{Im}\,(c_p) = 0$ (see Green (1970) for further discussion of this point). In obtaining reduction of the amplification rates, it would appear that removal of the upper rather than the lower parts of the stability curves is of far more importance.

To illustrate the effect of damping variation on stability we set $m = 1$, $c_W = 5000$, and compare the stability results for $D = 250$ and 1000. Figures 2 and 3 show the stability curves with the upper part of the stability curve for m, D, $c_0 = 1$, 250, 5000, respectively, plotted to high k (where the classical asymptotic analysis would be completely inadequate) to demonstrate that, in this case,

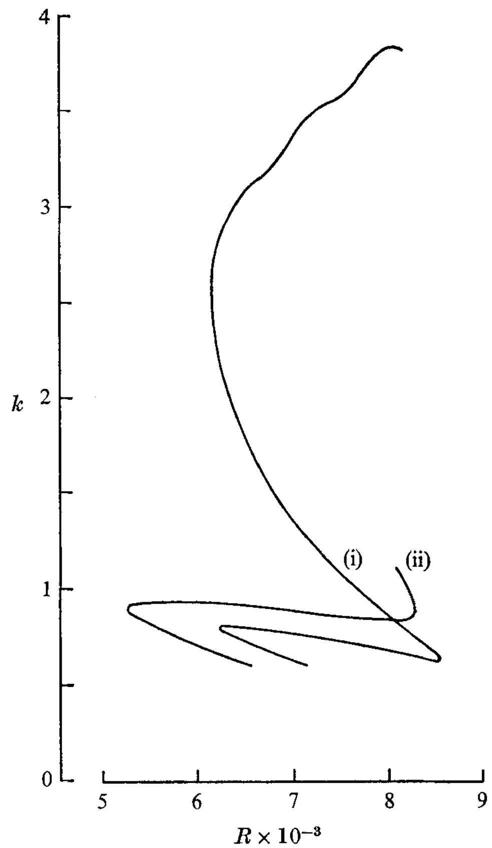

FIGURE 2. Neutral stability curves for damped membrane walls. Wave-number k versus Reynolds number R, for $m = 1$, $c_W = 5000$, $N = 30$.

	D	R_c	k_c
(i)	250	6190	2·60 (upper branch)
(ii)	1000	5270	0·89

the upper region determines the critical Reynolds number. It can be seen that the decrease in damping has stabilized the lower part of the stability curve while destabilizing the upper part. The waviness at high k is due to the accuracy of the computed results dropping off; a few points calculated for the stability curve with a thirty-five term series, instead of the usual thirty, showed that this waviness may be smoothed out. Figure 3 is presented to show that on the upper part of the neutral stability curve c_p reaches an almost constant value which is slightly less than c_W.

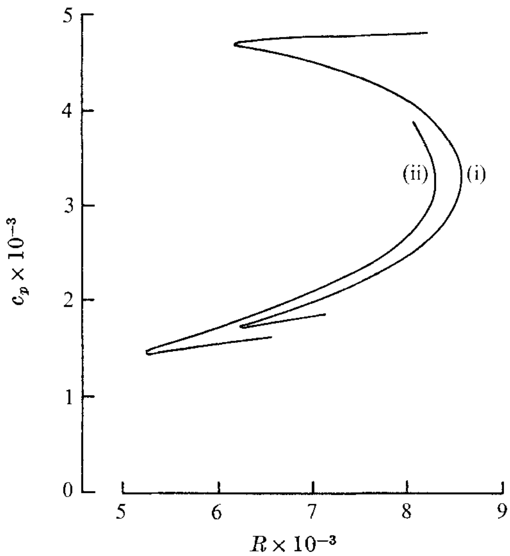

FIGURE 3. Spatial stability curves for damped membrane walls. Wave speed c_p versus Reynolds number R, conditions as in figure 2.

6. Conclusions

The results demonstrate that a modal solution based on a variational formulation of the boundary-value problem can be used to obtain accurate results for the stability of plane Poiseuille flow between flexible walls. We suggest, in the light of these results, that the method has application to a variety of similar problems.

It may be noted that for certain flexible wall situations it is possible to transform, quite readily, the neutral stability curve for rigid-wall plane Poiseuille flow into neutral stability curves for the flexible-wall problem, by use of the asymptotic analysis developed by Lin (1945, 1955). The basis of such a transformation lies in the analysis of Benjamin (1960) for the qualitative behaviour of the stability curves for the flexible-wall boundary-layer problem. For the present problem we take that analysis a stage further to establish a precise form for the transformation of the Tollmien–Schlichting stability curve for a rigid wall to that for a membrane wall with no structural damping. We start with the characteristic equation in the usual form and notation (see Benjamin (1960), where $\lambda = 0$, and Landahl (1962)):

$$u + iv + \alpha = \mathscr{F}(z)/[1+\lambda(1-\mathscr{F}(z))], \tag{13}$$

where, in our notation, for a membrane without structural damping,

$$\alpha = -cU'(-1)/mk^2(c_W^2/R^2 - c^2)$$

and (from Lin 1955, pp. 37, 38 and 40)

$$u = -\frac{v}{\pi}\log c + O(1) + \left\{cU'(-1)\Big/k^2\int_{-1}^{0}(U-c)^2\,dy\right\}(1+O(k^2)),$$

$$v = \{-\pi c U'(-1)\,U''(y_c)/U'(y_c)^3\}(1+O(c^3)),$$

$$\lambda = ((y_c+1)/c)\,U'(-1) - 1,$$

$\mathscr{F}(z)$ is the modified Tietjens function and

$$z = \{kRU'(y_c)\}^{\frac{1}{3}}(y_c+1).$$

These terms, as given, are to the order of accuracy employed in the calculations of the rigid-wall stability curve (Lin 1945), which is known to be reasonably accurate. Since λ and v are functions of c alone to this order, $\mathscr{F}$ is a function of z alone, z is a function of kR and c alone and α is real, the parameters k_0, R_0 and c, defining the rigid-wall neutral stability curve (corresponding to $\alpha = 0$), may be used to obtain the parameters k, R and c, defining the flexible-wall neutral stability curves, by keeping c and z constant and only changing u through k (see Benjamin 1960). Thus

$$u(k,c)+\alpha = u(k_0,c) \quad \text{and} \quad z(k,R,c) = z(k_0,R_0,c),$$

or

$$cU'(-1)\Big/k^2\int_{-1}^{0}(U-c)^2dy+\alpha = cU'(-1)\Big/k_0^2\int_{-1}^{0}(U-c)^2dy \quad \text{and} \quad kR = k_0R_0.$$

On substituting for U and α, and performing the integration, we have

$$(1/k^2)\{f(c)^{-1}-\{m(c_W^2/R^2-c^2)\}^{-1}\} = (1/k_0^2)f(c)^{-1} \tag{14}$$

and $kR = k_0R_0$, where $f(c) = \frac{8}{15}-\frac{4}{3}c+c^2$. By eliminating R and solving for k/k_0 we may draw a number of conclusions concerning the transformation of the rigid-wall curves. In particular, for membranes under tension only, writing $mc_W^2 = T$ and $m = 0$, we have

$$(k/k_0)^2 = \{\tfrac{1}{2}\pm(\tfrac{1}{4}-[R_0^2/T]f(c))^{\frac{1}{2}}\}. \tag{15}$$

The two roots of this equation imply that each point on the rigid-wall curve has a double image in the transformed plane. It may be further established that the neutral stability curve would be expected to close, in agreement with the Hains & Price conclusion (1962), since, for sufficiently large R_0, k would become complex. (This result is confined neither to inertialess membranes nor plane Poiseuille flow.) The neutral stability curve given by Lin (1945) for rigid-wall plane Poiseuille flow is used in the above transformation to give a neutral stability curve for the case when $T = 0{\cdot}5\times10^9$. This curve is compared in figure 4 with the neutral stability curve obtained from the present program and that of Hains & Price for the same value of T. It may be seen that agreement is good along the low R portions of the curves, while there is a marked disagreement between the curve of Hains & Price and the other two curves along the high R portions. (The curve for the present work has not been completed, since the program is unsuited to examine stability at the low values of k and high values of R involved.) Since, for the asymptotic analysis, the part of the rigid-wall stability curve needed in the transformation is of known accuracy (for a given R less than $1{\cdot}8\times10^4$ the error in k is less than 5 %) and since the same order of accuracy is employed in the transformation, we would expect the flexible-wall stability curve to be of the same accuracy. It is therefore surprising to find that the Hains & Price results are in such disagreement with the asymptotic theory and our calculations.

Proceeding a stage further we may deduce from (15) that at the value of T for which $\frac{1}{4}-R_0^2f(c)/T = 0$, when R_0 is the rigid-wall critical Reynolds number, the region of instability shrinks to a point and that for T less than this value there is no transformed stability curve. Inserting the values given by Lin (1945) for k_0, R_0 and c, corresponding to the rigid-wall critical Reynolds number, we find that the asymptotic theory predicts the curves closing to a point for $T = 2{\cdot}8 \times 10^7$, a value which is less than half that obtained by Hains & Price ($T = 6{\cdot}65 \times 10^7$). Further, the point at which the curves disappear is given as $R = 7500$, $k = 0{\cdot}74$ by the asymptotic theory and as $R = 8240$, $k = 0{\cdot}85$ by Hains & Price. The present program was used to search the neighbourhood of these points and, again, the asymptotic theory was found to be more accurate.

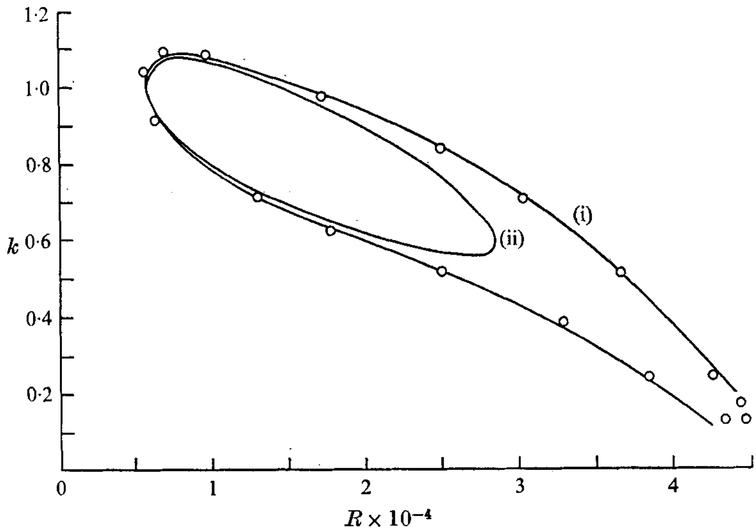

FIGURE 4. Neutral stability curves for membrane walls (tension only, $T = 0{\cdot}5 \times 10^9$). Wave-number k versus Reynolds number R for (i) present calculations ($N = 30$); (ii) Hains & Price (1962); ○, isolated points from asymptotic analysis.

The low wave-number region always causes some difficulty for the boundary-layer situation because the dominant term in the perturbation pressure is the inviscid pressure, which is to be expected when the boundary-layer thickness becomes insignificant on a wavelength scale, and the stability is then governed by the classical Kelvin–Helmholtz analysis. A pure membrane loses its stiffness for large wavelengths (because of the very small surface curvature) and is unstable when coupled to an inviscid flow. This has led to considerable discussion (Benjamin 1960; Landahl 1962) and the employment of 'local' stiffness terms in the membrane formulation (Landahl 1962; Kaplan 1964). Such a difficulty does not arise for plane Poiseuille flow since the coupling of the membranes to the flow does not necessarily cause instability in this case. To show this, the leading term from the asymptotic expression for the pressure must be derived (by use of the results of Lin (1955)) and then inserted into the membrane equation giving

$$c^2(1+m) - c(\tfrac{4}{3} - i\,Dm/kR) + (\tfrac{8}{15} - m\,c_W^2/R^2) = 0.$$

This equation is equivalent to that used by Landahl (1962) to illustrate the role of damping in the boundary-layer problem and it predicts that instability occurs, for $D \neq 0$, when

$$(1+m)(\tfrac{8}{15}-m\,c_W^2/R^2) > 0 \quad \text{or} \quad R > (\tfrac{15}{8}mc_W^2)^{\frac{1}{2}},$$

and, for $D = 0$, when

$$(1+m)(\tfrac{8}{15}-m\,c_W^2/R^2) > \tfrac{4}{9} \quad \text{or} \quad R > \{(1/mc_W^2)(\tfrac{8}{15}-4/9(1+m))\}^{-\frac{1}{2}}.$$

We conclude, therefore, that as k tends to zero the stability boundaries in the R, k plane intercept the R axis at finite non-zero R and that here also the effect of damping is destabilizing. Inserting the values of m and c_W considered in the present calculations in these formulae, we obtain Reynolds numbers which are greater than those for the noses of the respective flexible-wall curves. Hence the critical Reynolds number is in all cases determined by the computed portions of the curves.

We may finally remark on the approach employed in the present work. Keeping the c dependence of E implicit does not take full advantage of a modal type solution since the eigenvalue spectra are not directly related to a given physical wall (unless highly idealized). With further computing scope the present method of solution could be modified to solve directly for the eigenvalues. Noting that c occurs quadratically in the present problem, see §2 and especially equation (4), the retention of c in explicit form when setting up the matrix equations generates a matrix equation of the form $(\mathbf{A}c^2+\mathbf{B}c+\mathbf{D})\mathbf{x} = \mathbf{0}$, where $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are matrices of the same order as those used in the present method, c is the eigenvalue and $\mathbf{x}$ the eigenvector. This may be converted to standard form by substituting $\mathbf{I}c\mathbf{x} = \mathbf{I}\mathbf{x}_1$, where $\mathbf{I}$ is the unit matrix and $\mathbf{x}_1$ a vector, and solving as two simultaneous equations to give

$$\left\{\begin{pmatrix} \mathbf{0} & -\mathbf{I} \\ \mathbf{A}^{-1}\mathbf{D} & \mathbf{A}^{-1}\mathbf{B} \end{pmatrix} + c\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}\right\}\begin{Bmatrix} \mathbf{x} \\ \mathbf{x}_1 \end{Bmatrix} = \mathbf{0}.$$

In this form, the matrices are of twice the order of those involved in the present method. The great advantage is that this would provide the complete spectrum of eigenvalues of the *physical* problem directly and would eliminate the need for an iterative scheme, as used in the present solution.

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